ASYMPTOTIC APPROXIMATION TO THE SOLUTION OF THE NONLINEAR HEAT-CONDUCTION PROBLEM

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We have derived an approximate solution for the nonlinear problem of heat conduction for a composite moving region in the case of a variable velocity of motion for the medium.

In this article we examine the problem of temperature-field distribution for a moving medium consisting of two media exhibiting different thermophysical characteristics. The given region, containing the axis of abscissas, moves at a velocity which is a linear function of the temperature gradient. In each of these regions, the heat-conduction coefficient is an exponential function of temperature, i.e., $K(u) = K_0u^2$. Moreover, c = const and ρ = const.

The problem is described by the following systems of nonlinear equations:

$$\frac{\partial u_1}{\partial \tau} = \frac{\partial}{\partial x}\left[K_1(u_1)\frac{\partial u_1}{\partial x}\right] - \varepsilon_1\left[\left(\frac{\partial u_1}{\partial x}\right)^2 + u_1\frac{\partial^2 u_1}{\partial x^2}\right], \tag{1}$$

$$x \in (-\infty,\ 0];$$

$$\frac{\partial u_2}{\partial \tau} = \frac{\partial}{\partial x}\left[K_2(u_2)\frac{\partial u_2}{\partial x}\right] - \varepsilon_2\left[\left(\frac{\partial u_2}{\partial x}\right)^2 + u_2\frac{\partial^2 u_2}{\partial x^2}\right], \tag{2}$$

$$x \in [0,\ \infty),$$

for the following initial and boundary conditions:

$$u_1(x,\ 0) = u_{01},\quad u_2(x,\ 0) = u_{02},\quad u_1(0,\ \tau) = u_2(0,\ \tau); \tag{3}$$

$$K_1(u_1)\frac{\partial u_1(0,\ \tau)}{\partial x} = K_2(u_2)\frac{\partial u_2(0,\ \tau)}{\partial x}, \tag{4}$$

where ε_1, ε_2, u_{01}, and u_{02} are constants; $\varepsilon_1 = K_0u_0^2$ and $\delta_1 = \varepsilon u_0$.

With the assumptions made here, it is extremely difficult [1] to find an exact analytical solution for the sought system. In this paper we have, therefore, succeeded through the use of power expansions to expand the class of solved problems and to find an approximate solution within the class of holomorphic functions [2].

With a self-similar formulation, we achieve a system which is written in the following form:

$$-2t\frac{d\Psi_1}{dt} = \left(\frac{d\Psi_1}{dt}\right)^2(2\varepsilon_{01}\Psi_1 - \delta_{01}) + \frac{d^2\Psi_1}{dt^2}\Psi_1(\varepsilon_{01}\Psi_1 - \delta_{01}), \tag{5}$$

$$x \in (-\infty,\ 0];$$

$$-2t\frac{d\Psi_2}{dt} = \left(\frac{d\Psi_2}{dt}\right)^2(2\varepsilon_{02}\Psi_2 - \delta_{02}) + \frac{d^2\Psi_2}{dt^2}\Psi_2(\varepsilon_{02}\Psi_2 - \delta_{02}), \tag{6}$$

$$x \in [0,\ \infty).$$

The boundary conditions assume the form

$$\Psi_1|_{t=\infty} = \Psi_{01},\quad \Psi_2|_{t=\infty} = \Psi_{02}; \tag{7}$$

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$$\Psi_1|_{t=0} = \Psi_2|_{t=0}; \tag{8}$$

$$K_1(\Psi_1)\left(\frac{d\Psi_1}{dt}\right)_{t=0} = -K_2(\Psi_2)\left(\frac{d\Psi_2}{dt}\right)_{t=0}. \tag{9}$$

Here ε_{01}, ε_{02}, δ_{01}, and δ_{02} are constants, and t is a space variable.

Equations (5) and (6) with boundary conditions (7)-(9) with extremely general assumptions relative to the functions $K(\Psi)$ and $\Psi(t)$ have a uniform solution [3].

The solution for (5) and (6) is sought formally in the form of asymptotic power expansions:

$$\Psi_1(t) = \sum_{n=0}^{\infty} a_n(C_p)(t-t_1)^n, \tag{10}$$

$$\Psi_2(t) = \sum_{n=0}^{\infty} b_n(D_q)(t-t_2)^n. \tag{11}$$

In this case t_1 and t_2 are, respectively, arbitrary points in two regions; C_p and D_q are constants which are functions of the constants in the equations and of the geometric dimensions of the regions. Moreover, as will be demonstrated later on,

$$a_{n+\nu} = F_1(a_0, a_1, \ldots, a_{n+\nu-1}, b_0, b_1, \ldots, b_{n+\nu-1}, C_p, D_q), \tag{12}$$

$$b_{n+\nu} = F_2(b_0, b_1, \ldots, b_{n+\nu-1}, a_0, a_1, \ldots, a_{n+\nu-1}, C_p, D_q). \tag{13}$$

For the sake of convenience, dividing both parts of (5) and (6) by Ψ_1 and Ψ_2, respectively, we derive a more convenient form of notation.

Concepts similar to (10) and (11) permit all of the terms in (5) and (6).

We denote

$$\frac{1}{\Psi_1(t)} = \sum_{n=0}^{\infty} E_n(t-t_1)^n. \tag{14}$$

Then, taking the logarithms and the derivatives of the two parts of (14), we obtain

$$E_0 = a_0^{-1}, \quad E_n = -a_0 \sum_{l=1}^{k} a_l E_{k-l}^{(1)}.$$

Analogously, if

$$\frac{1}{\Psi_2(t)} = \sum_{n=0}^{\infty} H_n(t-t_2)^n,$$

we have

$$H_0 = b_0^{-1}, \quad H_n = -b_0 \sum_{l=1}^{k} b_l H_{k-l}^{(1)}. \tag{15}$$

The terms in the left-hand members of (5) and (6) permit the following representations with respect to powers of the space variable:

$$\frac{2t}{\Psi_1(t)}\frac{d\Psi_1(t)}{dt} = \frac{2t}{a_0}\sum_{n=0}^{\infty}(n+1)a_{n+1}(t-t_1)^n - \frac{2t}{a_0}\sum_{n=0}^{\infty}\sum_{k=1}^{n}(n+1-k)a_{n+1-k}\sum_{l=1}^{k}a_l E_{k-l}^{(1)}(t-t_1)^n, \tag{16}$$

$$\frac{2t}{\Psi_2(t)}\frac{d\Psi_2}{dt} = \frac{2t}{b_0}\sum_{n=0}^{\infty}(n+1)b_{n+1}(t-t_2)^n - \frac{2t}{b_0}\sum_{n=0}^{\infty}\sum_{k=1}^{n}(n+1-k)b_{n+1-k}\sum_{l=1}^{k}b_l H_{k-l}^{(1)}(t-t_2)^n. \tag{17}$$

Using the Cauchy formula and the exponential representations proposed in [2], we obtain the following expansions:

$$2\varepsilon_{01}\left(\frac{d\Psi_1}{dt}\right)^2 = 2\varepsilon_{01}\sum_{n=0}^{\infty}\sum_{m=1}^{n} m(n-m+1)a_m a_{n-m+1}(t-t_1)^n, \tag{18}$$

$$\varepsilon_{01}\Psi_1(t)\frac{d^2\Psi_1}{dt^2}=\varepsilon_{01}\sum_{n=0}^{\infty}\sum_{i=2}^{n+2}i(i-1)a_ia_{n+2-i}(t-t_1)^n, \tag{19}$$

$$\delta_{01}\frac{1}{\Psi_1(t)}\left(\frac{d\Psi_1}{dt}\right)^2=\delta_{01}\frac{a_1}{a_0}\sum_{n=0}^{\infty}(n+1)a_{n+1}(t-t_1)^n+\delta_{01}\sum_{n=0}^{\infty}\sum_{i=1}^{n}ia_i\gamma_{n+1-i}^{(2)}(t-t_1)^n, \tag{20}$$

in which we have introduced the notation

$$\begin{gathered}\gamma_0^{(2)}=a_1,\quad \gamma_n^{(2)}=(n+1)a_{n+1}-\sum_{j=1}^{n}(n+1-j)a_{n+1-j}\sum_{s=1}^{j}a_sE_{j-s}^{(1)},\\ \delta_{01}\frac{d^2\Psi_1}{dt^2}=\delta_{01}\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(t-t_1)^n,\end{gathered} \tag{21}$$

Similar power expansions, under the same assumptions, are permissible for the terms in [6].

Given sufficiently general assumptions with respect to $K(\Psi)$ and $\Psi(t)$ for both of the regions, from the condition of conjugacy at the point $t = 0$ we derive the following representations:

$$\varepsilon_{01}\sum_{n=0}^{\infty}\sum_{h=0}^{n}(h+1)a_{h+1}\sum_{\omega=0}^{n-h}a_\omega a_{n-\omega}(t-t_1)^n=-\varepsilon_{02}\sum_{n=0}^{\infty}\sum_{z=0}^{n}(z+1)b_{z+1}\sum_{\chi=0}^{n-z}b_\chi b_{n-\chi}(t-t_2)^n \tag{22}$$

and the same thing holds for the conjugacy conditions (8) and the boundary conditions (7).

Let us substitute all of the derived representations into (5) and (6).

From the boundary condition (7) we find the first coefficients of the unknown functions Ψ_1 and Ψ_2:

$$a_0=\Psi_{01},\quad b_0=\Psi_{02}. \tag{23}$$

For convenience in the subsequent analysis of the solution and for the sake of generality in calculation, we will assume that $t_2 = \eta t_1$, where $\eta > 1$.

For the conjugacy condition (22) and from the first conjugacy condition we find a_1 and b_1:

$$a_1=\frac{b_0^2(b_0-a_0)}{a_0^2\eta-b_0},\quad b_1=\frac{a_0^2(b_0-a_0)}{b_0^2-a_0^2\eta}. \tag{24}$$

From the exponential representations of (5) and (6) for $n = 0$ and from the conjugacy condition for $n = 2$ we find

$$\begin{gathered}a_2=\frac{N_0(\delta_{02}-\varepsilon_{02}b_0)-M_0(\eta^2+\varepsilon_{02}b_0b_1)}{W_0},\\ b_2=\frac{N_0(\varepsilon_{01}a_0-\delta_{01})-M_0(1+\varepsilon_{01}a_0a_1)}{W_0},\end{gathered} \tag{25}$$

where

$$\begin{gathered}M_0=\frac{a_1(\delta_{01}a_1-2)}{2a_0}+\frac{b_1(\delta_{02}b_2-2)}{2b_0},\\ N_0=\varepsilon_{02}b_0b_1^2-\varepsilon_{01}a_0a_1^2,\\ W_0=-\left[(\varepsilon_{01}a_0-\delta_{01})(\eta^2+\varepsilon_{02}b_0b_1)+(1+\varepsilon_{01}a_0a_1)(\varepsilon_{02}b_0-\delta_{02})\right].\end{gathered}$$

Having transformed the expression for the coefficients in the second conjugacy condition to the form

$$(n+2)a_{n+2}a_0a_{n+1}-\eta^{n+1}(n+2)b_{n+2}b_0b_{n+1}=\eta^{n+1}\sum_{z=0}^{n}(z+1)b_{z+1}\sum_{\chi=0}^{n-z}b_\chi b_{n-\chi}-\sum_{h=0}^{n}(h+1)a_{h+1}\sum_{\omega=0}^{n-h}a_\omega a_{n-\omega}, \tag{26}$$

with condition (8) and the exponential representations of (5) and (6) we find the recursion formula for the determination of the coefficients in the unknown series (10) and (11):

$$a_{n+2}=\frac{N_n(\delta_{02}-\varepsilon_{02}b_0)(n+1)(n+2)+M_n\left[\eta^{n+2}+(n+2)b_0b_{n+1}\right]}{W_n}, \tag{27}$$

$$b_{n+2} = \frac{M_n\left[1 + (n+2)\,a_0 a_{n+1}\right] + N_n\,(\delta_{01} - \varepsilon_{01} a_0)\,(n+1)\,(n+2)}{W_n}, \tag{28}$$

where we have introduced the following notation:

$$M_n = M_{n,1} + M_{n,2};$$

$$M_{n,1} = \frac{2}{a_0}\left[(n+1)\,a_{n+1} - \sum_{k=1}^{n}(n+1-k)\,a_{n+1-k}\sum_{l=1}^{k} a_l E_{k-l}^{(1)}\right]$$

$$+ 2\varepsilon_{01}\sum_{m=1}^{n} m\,(n-m+1)\,a_m a_{n-m+1} + \varepsilon_{01}\sum_{i=2}^{n+1} i a_i\,(i-1)\,a_{n+2-i} - \frac{\delta_{01}}{a_0}\left[a_1\,(n+1)\,a_{n+1} + \sum_{i=1}^{n} i a_i \gamma_{n+1-i}^{(2)}\right];$$

$$M_{n,2} = \frac{2}{b_0}\left[(n+1)\,b_{n+1} - \sum_{k=1}^{n}(n+1-k)\,b_{n+1-k}\sum_{l=1}^{k} b_l H_{k-l}^{(1)}\right]$$

$$+ 2\varepsilon_{02}\sum_{m=1}^{n} m\,(n-m+1)\,b_m b_{n-m+1} + \varepsilon_{02}\sum_{i=2}^{n+1} i b_i\,(i-1)\,b_{n+2-i} - \frac{\delta_{02}}{b_0}\left[b_1\,(n+1)\,b_{n+1} + \sum_{i=1}^{n} i b_i \Theta_{n+1-i}^{(2)}\right];$$

$$\Theta_{n+1-i}^{(2)} = (n+1)\,b_{n+1} - \sum_{j=1}^{n}(n+1-j)\,b_{n+1-j}\sum_{s=1}^{j} a_s H_{j-s}^{(1)};$$

$$W_n = \left[1 + (n+2)\,a_0 a_{n+1}\right](\delta_{02} - \varepsilon_{02} b_0)\,(n+1)\,(n+2) + (\delta_{01} - \varepsilon_{01} a_0)\,(n+1)\,(n+2)\left[\eta^{n+2} + (n+2)\,b_0 b_{n+1}\right].$$

Under the condition that the coefficients of the equations in this system satisfy the condition for the existence of $\lim\limits_{n\to\infty} a_n/a_{n+p}$ and $\lim\limits_{n\to\infty} b_n/b_{n+q}$, the radius of convergence for series (10) and (11) can be determined numerically, by calculating a sufficient number of terms in the sequence

$$R_{n,1}^{p} = \frac{a_n}{a_{n+p}}, \quad R_{n,2}^{q} = \frac{b_n}{b_{n+q}} \tag{29}$$

to include the value from which the following equations will be satisfied:

$$\frac{a_n}{a_{n+p}} = \frac{a_{n+p}}{a_{n+2p}} = \ldots = \text{const} = R^p, \tag{30}$$

$$\frac{b_n}{b_{n+q}} = \frac{b_{n+q}}{b_{n+2q}} = \ldots = \text{const} = R^q \tag{31}$$

and with the accuracy that is required for the given problem.

As was pointed out above, this expansion is unique and has a derivative at some $t \geq t_0$, where t_0 = const. In this case there exists a constant ζ for which

$$|\Psi(t)| < \exp\frac{t^{\zeta+1}}{\zeta+1}. \tag{32}$$

With this method we can determine the analytical continuation of series (10) and (11) and find the singularities of the resulting solution. Regardless of the number of analytical continuations which permit improvement in the convergence of the expansions, the form of (27) and (28) remains unchanged, which is particularly convenient when using computers.

This method also makes it possible to construct relationships similar to (27) and (28) for any function K(u).

The control calculations based on the expansion algorithms were carried out on an M-20 computer.

LITERATURE CITED

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